

On the non-uniform motion of dislocations: The retarded elastic fields, the retarded dislocation tensor potentials and the Liénard-Wiechert tensor potentials

Markus Lazar ^{a,b,*}

^a Heisenberg Research Group,
Department of Physics,
Darmstadt University of Technology,
Hochschulstr. 6,
D-64289 Darmstadt, Germany

^b Department of Physics,
Michigan Technological University,
Houghton, MI 49931, USA

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Abstract

The purpose of this paper is the fundamental theory of the non-uniform motion of dislocations in two and three space-dimensions. We investigate the non-uniform motion of an arbitrary distribution of dislocations, a dislocation loop and straight dislocations in infinite media using the theory of incompatible elastodynamics. The equations of motion are derived for non-uniformly moving dislocations. The retarded elastic fields produced by a distribution of dislocations and the retarded dislocation tensor potentials are determined. New fundamental key-formulae for the dynamics of dislocations are derived (Jefimenko type and Heaviside-Feynman type equations of dislocations). In addition, exact closed-form solutions of the elastic fields produced by a dislocation loop are calculated as retarded line integral expressions for subsonic motion. The fields of the elastic velocity and elastic distortion surrounding the arbitrarily moving dislocation loop are given explicitly in terms of the so-called three-dimensional elastodynamic Liénard-Wiechert tensor potentials. The two-dimensional elastodynamic Liénard-Wiechert tensor potentials and the near-field approximation of the elastic fields for straight dislocations are

**E-mail address:* lazar@fkp.tu-darmstadt.de (M. Lazar).

calculated. The singularities of the near-fields of accelerating screw and edge dislocations are determined.

Keywords: dislocation dynamics; non-uniform motion; dislocation loop; elastodynamics; radiation; retarded fields; near-fields; singularities.

1 Introduction

The investigation of the non-uniform motion of dislocations is an important interdisciplinary research field. It has attracted the attention of scientists from several different fields such as applied mathematics, solid state physics, material science, continuum mechanics, and seismology (see, e.g., [1, 2, 3]). Typical problems of elastodynamics of dislocations are the determination of elastic fields produced by the non-uniform motion of straight dislocations and dislocation loops.

The theory of the non-uniform motion of Volterra dislocations has a long history starting with the famous and well-known papers of Eshelby [4, 5]. Using an electromagnetic analogy, Eshelby [5] found the elastic fields of a non-uniformly moving screw dislocation. The problem of the non-uniform motion of a gliding edge dislocation was solved by Kiusalaas and Mura [6]. Kiusalaas and Mura [6] gave the solution in terms of ‘stress (or potential) functions’ for the velocity and elastic distortion fields (see also [7, 1]). Lazar [8] has written a systematic review paper about the non-uniform motion of straight dislocations including the solution of a non-uniformly climbing edge dislocation.

The behaviour of a straight dislocation is somehow particular, because at any time the fields are determined not only by the instantaneous values, but also by the values in the past [4, 5]. As Eshelby [4] succinctly put it: ‘The dislocation is haunted by its past’. For that reason, all the solutions of the elastic fields of non-uniformly moving straight dislocations are given in the form as time integrals and show an afterglow. Due to the afterglow, Huygens’ principle is not valid in two dimensions (see, e.g., [9, 10, 11]). In general, a two-dimensional wave-motion possesses a ‘tail’. The wave motion in two-dimensions and the afterglow effect are discussed more in detail by Baker and Copson [12], Barton [13], and Lazar [8].

The non-uniform motion of dislocation loops was also studied. The solution of non-uniformly moving dislocation loops was first formulated by Mura [14] in terms of the three-dimensional elastodynamic Green tensor as double integrals over the loop curve and time (see also [15]). The mathematical formulation of moving dislocation loops and analogous double integral presentations were also given by Kossecka [16] and Kossecka and deWit [17, 18]. It was pointed out by Markenscoff [19] (see also [20, 21]) that some care is necessary in the calculation of the elastic fields produced by non-uniformly moving dislocations in order to avoid non-integrable singularities. Markenscoff [19] showed that the general expressions for the velocity and elastic distortion fields of dislocations given by Mura [14, 15] are not free of non-integrable singularities. The reason is that the integration and differentiation cannot be changed in some cases.

In general, the problem of moving dislocation loops is a three-dimensional problem in space. It is well-known that Huygens’ principle is only valid for odd space dimensions:

$N \geq 3$ [9, 10, 11]. For that reason, there is no afterglow in three dimensions. In standard books on electrodynamics (see, e.g., [22, 23]), the electric scalar and magnetic vector potentials of a non-uniformly moving point charge which are the famous Liénard-Wiechert potentials [24, 25] can be found. It is quite surprising that nothing has been done in this direction in the elastodynamics of moving dislocation loops up to now. In elastodynamics, only the retarded potentials were given for the waves produced by body forces, using the Helmholtz decomposition (see, e.g., [10, 26]). A more general expression for the retarded potential in elastodynamics was given by Hudson [27]. Though Emil Wiechert was a director of the geophysical laboratory in Göttingen, he formulated his theory for electrodynamics [25], but not for elastodynamics, with which he was certainly familiar. In this paper, we investigate some fundamental problems of the elastodynamical theory of dislocations in analogy to the electromagnetic field theory. Especially, we want to develop the Liénard-Wiechert potentials for the elastodynamics and to apply them to dislocation loops and straight dislocations.

This paper is organized as follows. In Section 2, the framework of incompatible elastodynamics and the equations of motion of dislocations are presented. In Section 3, the equations of motion for an arbitrary three-dimensional distribution of dislocations are solved, using the three-dimensional elastodynamic Green tensor. As a result the retarded elastic fields are given. The non-uniform motion of a closed dislocation loop is studied in Section 4. The so-called elastodynamic Liénard-Wiechert tensor potentials are determined. The elastic fields are given in terms of so-called Liénard-Wiechert tensor potentials. The static limit of the elastic fields of the non-uniformly moving dislocation loop is given in Section 5. In Section 6, the two-dimensional elastodynamic Liénard-Wiechert tensor potentials and the elastic fields of non-uniformly moving straight dislocations are calculated. In Section 7, the near-field approximation of accelerating screw and edge dislocations are calculated. In a straightforward manner, the $1/R$ -singularity and a logarithmic singularity associated with the acceleration of the dislocation are found. The relation between the elastodynamic Liénard-Wiechert tensor potentials and Mura's dislocation tensor potentials is presented in Section 8. The retarded dislocation tensor potentials and the proper elastodynamic Liénard-Wiechert tensor potentials of a dislocation loop are determined. In Section 9, the conclusions are given.

2 The equations of motion of dislocations

In this section, we derive the equations of motion for dislocations in the framework of incompatible elastodynamics (see, e.g., [14, 15, 16, 28, 29, 8]). An unbounded, isotropic, homogeneous, linearly elastic solid is considered. In the elasticity theory of self-stresses the equilibrium condition is¹

$$\dot{p}_i = \sigma_{ij,j}, \quad (1)$$

where \mathbf{p} and $\boldsymbol{\sigma}$ are the linear momentum vector and the force stress tensor, respectively. In the incompatible linear elasticity, the momentum vector \mathbf{p} and the stress tensor $\boldsymbol{\sigma}$ can

¹We use the usual notation $\beta_{ij,k} := \partial_k \beta_{ij}$ and $\dot{\beta}_{ij} := \partial_t \beta_{ij}$.

be expressed in terms of the incompatible elastic velocity (particle velocity) vector \mathbf{v} and the incompatible elastic distortion tensor $\boldsymbol{\beta}$ by means of the two constitutive relations

$$p_i = \rho v_i, \quad (2)$$

$$\sigma_{ij} = C_{ijkl} \beta_{kl}, \quad (3)$$

where ρ denotes the mass density and C_{ijkl} is the tensor of elastic moduli. The tensor C_{ijkl} is characterized by the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \quad (4)$$

If the constitutive relations (2) and (3) are substituted in Eq. (1), the equilibrium condition expressed in terms of the elastic fields \mathbf{v} and $\boldsymbol{\beta}$ is obtained

$$\rho \dot{v}_i = C_{ijkl} \beta_{kl,j}. \quad (5)$$

The presence of dislocations makes the elastic fields incompatible which means that they are not anymore simple gradients or time derivatives of a displacement vector \mathbf{u} . Unlike the displacement field and the plastic fields, the elastic fields are physical state quantities of dislocations. This is one reason, why we deal only with the calculation of the elastic fields in this paper.

Other important tensor fields of dislocations are the dislocation density and dislocation current tensors (e.g. [28, 30]). The dislocation density tensor \mathbf{T} and the dislocation current tensor \mathbf{I} are defined by (see also [31, 30])

$$T_{ijk} = \beta_{ik,j} - \beta_{ij,k}, \quad (6)$$

$$I_{ij} = \dot{\beta}_{ij} - v_{i,j}. \quad (7)$$

The dislocation current tensor \mathbf{I} was originally introduced by Kosevich [32, 33] and Holländer [34, 35] (see also [28, 29, 36, 37]). The dislocation current tensor (7) is the difference of two pieces: the time derivative of the elastic distortion, and the elastic velocity gradient. It is noted that $T_{ijk} = -T_{ikj}$. Both \mathbf{T} and \mathbf{I} have nine independent components. A field theoretical justification for the structure of \mathbf{T} and \mathbf{I} was given by Lazar and Anastassiadis [38] (see also [30]). Moreover, they fulfill the Bianchi identities (see also [39, 40])

$$\epsilon_{jkl} T_{ijk,l} = 0, \quad (8)$$

$$\dot{T}_{ijk} + I_{ij,k} - I_{ik,j} = 0, \quad (9)$$

which are ‘conservation’ laws. Here ϵ_{jkl} denotes the Levi-Civita tensor. Eq. (8) states that dislocations cannot end inside the medium and Eq. (9) means that the time evolution of the dislocation density tensor \mathbf{T} is determined by the ‘curl’ of the dislocation current tensor \mathbf{I} . The tensors \mathbf{T} and \mathbf{I} may describe single straight dislocations, dislocation loops and an arbitrary distribution of dislocations.

Alternatively, we may rewrite the tensor \mathbf{T} as (dual) tensor of rank two

$$\alpha_{ij} = \frac{1}{2} \epsilon_{jkl} T_{ikl} = \epsilon_{jkl} \beta_{il,k} \quad (10)$$

with the inverse relation

$$T_{ikl} = \epsilon_{jkl} \alpha_{ij}. \quad (11)$$

The tensor α is the usual dislocation density tensor². Then the Bianchi identities (conservation laws) (8) and (9) simplify to (see also [28, 44, 29, 45, 36])

$$\alpha_{ij,j} = 0, \quad (12)$$

$$\dot{\alpha}_{ij} + \epsilon_{jkl} I_{ik,l} = 0. \quad (13)$$

Now we derive separated equations for the elastic fields β and \mathbf{v} as equations of motion. If we differentiate the Eq. (5) with respect to x_m and use Eqs. (6) and (7) to eliminate \mathbf{v} , we get the equation of motion for the incompatible elastic distortion tensor β (see also [16, 36, 8])

$$\rho \ddot{\beta}_{im} - C_{ijkl} \beta_{km,jl} = C_{ijkl} T_{kml,j} + \rho \dot{I}_{im}, \quad (14)$$

where the dislocation density and the dislocation current tensors are the sources. Eq. (14) is a tensorial Navier equation for β . Similarly, if we perform the differentiation of Eq. (5) with respect to time and use Eq. (7) to eliminate β , we obtain the equation of motion for the incompatible elastic velocity vector \mathbf{v} (see also [16, 36, 28, 8])

$$\rho \ddot{v}_i - C_{ijkl} v_{k,jl} = C_{ijkl} I_{kl,j}, \quad (15)$$

where the dislocation current tensor is the source term. Eq. (15) is a vectorial Navier equation for \mathbf{v} . We want to note that also Rogula [46] obtained equations of the form (14) and (15) for dislocations in a pseudo-continuum

3 The retarded elastic fields

In this section, we calculate the retarded elastic fields produced by a three-dimensional distribution of dislocations. The solutions of Eqs. (14) and (15) can be represented as convolution integrals [14, 16, 18, 8]. For an unbounded medium and under the assumption of zero initial conditions, which means that $\beta(\mathbf{r}, t_0)$ and $\mathbf{v}(\mathbf{r}, t_0)$ and their first time derivatives are zero for $t_0 \rightarrow -\infty$, the solutions of β and \mathbf{v} can be represented as

$$\begin{aligned} \beta_{im}(\mathbf{r}, t) &= \partial_k \int_{-\infty}^t \int_{-\infty}^{\infty} C_{jklm} G_{ij}(\mathbf{r} - \mathbf{r}', t - t') T_{lmn}(\mathbf{r}', t') d\mathbf{r}' dt' \\ &\quad + \partial_t \int_{-\infty}^t \int_{-\infty}^{\infty} \rho G_{ij}(\mathbf{r} - \mathbf{r}', t - t') I_{jm}(\mathbf{r}', t') d\mathbf{r}' dt' \end{aligned} \quad (16)$$

²In the literature, the notations of the dislocation density tensor and the dislocation current tensor are not unique: α (Lazar [30]) = α (Kossecka [16, 41]) = α^T (Kossecka and deWit [17]) = α^T (Kröner [42]) = α^T (deWit [43]) = $-\alpha$ (Teodosiu [36]) = $-\alpha^T$ (Kosevich [28]) and \mathbf{I} (Lazar [30]) = \mathbf{I} (Kossecka [16]) = $-\mathbf{I}^T$ (Kossecka and deWit [17]) = $-\mathbf{I}$ (Teodosiu [36]) = \mathbf{I}^T (Kosevich [28]).

and

$$v_i(\mathbf{r}, t) = \partial_k \int_{-\infty}^t \int_{-\infty}^{\infty} C_{jklm} G_{ij}(\mathbf{r} - \mathbf{r}', t - t') I_{lm}(\mathbf{r}', t') d\mathbf{r}' dt'. \quad (17)$$

It is obvious that dislocations act as a source of the elastodynamic fields. A proof that Eqs. (16) and (17) fulfill Eq. (5) can be found in Kossecka [44]. Here, G_{ij} is the elastodynamic Green tensor of the anisotropic Navier equation defined by

$$[\delta_{ik} \rho \partial_{tt} - C_{ijkl} \partial_j \partial_l] G_{km} = \delta_{im} \delta(t) \delta(\mathbf{r}), \quad (18)$$

where $\delta(\cdot)$ denotes the Dirac delta function and δ_{ij} is the Kronecker delta. For isotropic materials, the tensor of elastic moduli reduces to

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (19)$$

where λ and μ are the Lamé constants. Substituting Eq. (19) in Eq. (18), the isotropic Navier equation for the dynamic Green tensor was obtained

$$[\delta_{ik} \rho \partial_{tt} - \delta_{ik} \mu \Delta - (\lambda + \mu) \partial_i \partial_k] G_{km} = \delta_{im} \delta(t) \delta(\mathbf{r}), \quad (20)$$

where Δ denotes the Laplacian. When the material is isotropic and infinitely extended, the three-dimensional elastodynamic Green tensor reads [47, 48, 2, 3]

$$G_{ij}(\mathbf{r}, t) = \frac{1}{4\pi\rho} \left\{ \frac{\delta_{ij}}{rc_T^2} \delta(t - r/c_T) + \frac{x_i x_j}{r^3} \left(\frac{1}{c_L^2} \delta(t - r/c_L) - \frac{1}{c_T^2} \delta(t - r/c_T) \right) + \left(\frac{3x_i x_j}{r^2} - \delta_{ij} \right) \frac{1}{r^3} \int_{r/c_L}^{r/c_T} \tau \delta(t - \tau) d\tau \right\}, \quad (21)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. It should be pointed out that Eq. (21) is the retarded Green tensor. Here, c_L and c_T denote the velocities of the longitudinal and transversal elastic waves (sometimes called P- and S-waves). The two sound velocities can be given in terms of the Lamé constants ($c_T < c_L$)

$$c_L = \sqrt{\frac{2\mu + \lambda}{\rho}}, \quad c_T = \sqrt{\frac{\mu}{\rho}}. \quad (22)$$

The elastodynamic Green tensor (21) is a tensor-valued distribution with support along the two sound cones $r = c_T t$ and $r = c_L t$ as well as in the region between them. Eq. (21) consists of near-field and far-field terms. The first terms in (21) decay as $1/r$ and thus, they are the far-field terms. The last term in (21) decays more rapidly like $1/r^2$ which gives the near-field term (see, e.g., [3]).

If we use the relation

$$\frac{1}{r^2} \int_{r/c_L}^{r/c_T} \tau \delta(t - \tau) d\tau = \int_{1/c_L}^{1/c_T} \kappa \delta(t - \kappa r) d\kappa, \quad (23)$$

substitute the Green tensor (21) into Eqs. (16) and (17) and perform the integration in t' , the retarded elastic fields produced by an arbitrary three-dimensional distribution of dislocations are found

$$\begin{aligned}\beta_{im}(\mathbf{r}, t) = & \frac{1}{4\pi\rho} \partial_k \int_{\mathcal{V}} C_{jklm} \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) T_{lmn}(\mathbf{r}', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} T_{lmn}(\mathbf{r}', t_L) \right. \\ & \left. + \left(\frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa T_{lmn}(\mathbf{r}', t_\kappa) d\kappa \right\} d\mathbf{r}' \\ & + \frac{1}{4\pi} \partial_t \int_{\mathcal{V}} \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) I_{jm}(\mathbf{r}', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} I_{jm}(\mathbf{r}', t_L) \right. \\ & \left. + \left(\frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa I_{jm}(\mathbf{r}', t_\kappa) d\kappa \right\} d\mathbf{r}'\end{aligned}\quad (24)$$

and

$$\begin{aligned}v_i(\mathbf{r}, t) = & \frac{1}{4\pi\rho} \partial_k \int_{\mathcal{V}} C_{jklm} \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) I_{lm}(\mathbf{r}', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} I_{lm}(\mathbf{r}', t_L) \right. \\ & \left. + \left(\frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa I_{lm}(\mathbf{r}', t_\kappa) d\kappa \right\} d\mathbf{r}',\end{aligned}\quad (25)$$

where the so-called retarded times are given by

$$t_T = t - \frac{R}{c_T}, \quad (26)$$

$$t_L = t - \frac{R}{c_L}, \quad (27)$$

$$t_\kappa = t - \kappa R, \quad (28)$$

and κ is a dummy variable with the dimension $1/[\text{velocity}]$. Here t_T and t_L are the transversal retarded time and the longitudinal retarded time, respectively. The retarded time t_κ is an effective retarded time for the κ -integration with the limits $(1/c_L, 1/c_T)$. Since $c_L > c_T$, the retarded times fulfill: $t_T > t_L$ and $t_\kappa \in [t_L, t_T]$. Because the integrals (24) and (25) are evaluated at the retarded times, they are called retarded elastic fields. Here $R = |\mathbf{r} - \mathbf{r}'|$ is the distance from the source point \mathbf{r}' to the field point \mathbf{r} and is independent of t . The retarded elastic fields (24) and (25) are just integrals in \mathbf{r}' . Here \mathcal{V} is a volume integral in \mathbb{R}^3 . The sources \mathbf{T} and \mathbf{I} at the position \mathbf{r}' depend on the retarded times. The retarded elastic fields at the position \mathbf{r} and time t contain contributions from the past sound cones. Elastodynamic fields and waves propagate with finite velocities. Thus, there always is a time delay before a change in elastodynamic conditions initiated at a point of space can produce an effect at any other point of space. This time delay is called elastodynamic retardation. The retarded elastic fields (24) and (25) consist of three characteristic parts. The first term is the transversal one, transmitting with speed c_T , and it corresponds to S -wave motion. The second term is the longitudinal one, transmitting with speed c_L , and it corresponds to P -wave motion. Finally, the third term is neither longitudinal nor transversal and it gives contribution arriving at the speeds between the

two characteristic ones, which shows that this factor represents a combination of P -wave and S -wave motion. For a three-dimensional distribution of dislocations, we conclude that dislocations are retarded but not haunted by its past in contrast to a straight dislocation.

Now we carry out the differentiations in Eqs. (24) and (25) and use the relations

$$\begin{aligned} \partial_k T_{lmn}(\mathbf{r}', t_{\text{ret}}) &= -\frac{R_k}{cR} \partial_t T_{lmn}(\mathbf{r}', t_{\text{ret}}), \quad \partial_k I_{lm}(\mathbf{r}', t_{\text{ret}}) = -\frac{R_k}{cR} \partial_t I_{lm}(\mathbf{r}', t_{\text{ret}}), \\ \text{with } t_{\text{ret}} &= t_T, t_L, t_\kappa, \quad c = c_T, c_L, 1/\kappa, \end{aligned} \quad (29)$$

in order to find the retarded elastic field more explicitly. We obtain

$$\begin{aligned} \beta_{im}(\mathbf{r}, t) &= -\frac{1}{4\pi\rho} C_{jklm} \int_V \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j}{R^3} - \frac{3R_i R_j R_k}{R^5} \right) T_{lmn}(\mathbf{r}', t_T) \right. \\ &\quad + \frac{1}{c_T^3} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{R_k}{R^2} \partial_t T_{lmn}(\mathbf{r}', t_T) - \frac{1}{c_L^2} \left(\frac{\delta_{jk} R_i + \delta_{ik} R_j}{R^3} - \frac{3R_i R_j R_k}{R^5} \right) T_{lmn}(\mathbf{r}', t_L) \\ &\quad + \frac{1}{c_L^3} \frac{R_i R_j R_k}{R^4} \partial_t T_{lmn}(\mathbf{r}', t_L) - \left(\frac{\delta_{ij} R_k + 3\delta_{jk} R_i + 3\delta_{ik} R_j}{R^3} - \frac{9R_i R_j R_k}{R^5} \right) \int_{1/c_L}^{1/c_T} \kappa T_{lmn}(\mathbf{r}', t_\kappa) d\kappa \\ &\quad + \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{R_k}{R^2} \int_{1/c_L}^{1/c_T} \kappa^2 \partial_t T_{lmn}(\mathbf{r}', t_\kappa) d\kappa \Big\} d\mathbf{r}' \\ &\quad + \frac{1}{4\pi} \int_V \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) \partial_t I_{jm}(\mathbf{r}', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} \partial_t I_{jm}(\mathbf{r}', t_L) \right. \\ &\quad + \left. \left(\frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa \partial_t I_{jm}(\mathbf{r}', t_\kappa) d\kappa \right\} d\mathbf{r}' \end{aligned} \quad (30)$$

and

$$\begin{aligned} v_i(\mathbf{r}, t) &= -\frac{1}{4\pi\rho} C_{jklm} \int_V \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j}{R^3} - \frac{3R_i R_j R_k}{R^5} \right) I_{lm}(\mathbf{r}', t_T) \right. \\ &\quad + \frac{1}{c_T^3} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{R_k}{R^2} \partial_t I_{lm}(\mathbf{r}', t_T) - \frac{1}{c_L^2} \left(\frac{\delta_{jk} R_i + \delta_{ik} R_j}{R^3} - \frac{3R_i R_j R_k}{R^5} \right) I_{lm}(\mathbf{r}', t_L) \\ &\quad + \frac{1}{c_L^3} \frac{R_i R_j R_k}{R^4} \partial_t I_{lm}(\mathbf{r}', t_L) - \left(\frac{\delta_{ij} R_k + 3\delta_{jk} R_i + 3\delta_{ik} R_j}{R^3} - \frac{9R_i R_j R_k}{R^5} \right) \int_{1/c_L}^{1/c_T} \kappa I_{lm}(\mathbf{r}', t_\kappa) d\kappa \\ &\quad + \left. \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{R_k}{R^2} \int_{1/c_L}^{1/c_T} \kappa^2 \partial_t I_{lm}(\mathbf{r}', t_\kappa) d\kappa \right\} d\mathbf{r}'. \end{aligned} \quad (31)$$

These are the general retarded elastic fields produced by a time-dependent dislocation distribution. We now see from Eqs. (30) and (31) that the elastic distortion has three sources: the dislocation density \mathbf{T} , the time derivative of \mathbf{T} , and the time derivative of \mathbf{I} . And we see that the elastic velocity has two sources: the dislocation current \mathbf{I} , and the time derivative of \mathbf{I} . All of these sources are retarded due to the retarded times. Equations (30) and (31) show that, since both equations contain the time derivative of \mathbf{I} , the elastic distortion and the elastic velocity fields are created by the same time-variable dislocation current depending on the retarded times. Thus, the time-changing of the dislocation current \mathbf{I} is the common source for $\boldsymbol{\beta}$ and \mathbf{v} . The three δ_{ij} -terms multiplied

by $1/c_T$ -factors in Eq. (30) and the two δ_{ij} -terms multiplied by $1/c_T$ -factors in Eq. (31) are analogous to the Jefimenko formulae for the electromagnetic field strengths (see [49, 50, 23, 51]). Originally, Jefimenko [49] (see also [50]) derived the proper time-dependent generalizations of the Coulomb law and the Biot-Savart law as causal solutions of the Maxwell equations³. It is obvious that the retarded elastic fields (30) and (31) are more complicated than the Jefimenko equations in electrodynamics due to the tensor structure of the elastodynamical Green tensor and the dislocation-source fields. Nevertheless, we may call Eqs. (30) and (31) the Jefimenko type equations of dislocations. The causal dependencies of the elastodynamic phenomena of the motion of dislocations are described by Eqs. (30) and (31) which are exact solutions of the Navier equations (14) and (15) involving integrals over retarded dislocation sources.

Performing the κ -integration and using Eqs. (19) and (22) the static limit of Eq. (30), which gives the (static) dislocation version of the Biot-Savart law is calculated as

$$\beta_{im}(\mathbf{r}) = -\frac{1}{8\pi(1-\nu)} \int_V \left((1-2\nu) \frac{\delta_{il}R_n + \delta_{in}R_l - \delta_{ln}R_i}{R^3} + \frac{3R_iR_lR_n}{R^5} \right) T_{lmn}(\mathbf{r}') d\mathbf{r}'. \quad (32)$$

It gives the correct expression for the elastic distortion tensor in terms of the dislocation density tensor \mathbf{T} . Eq. (32) is the explicite expression of the isotropic version of the so-called Mura-Willis formula (see, e.g., [43, 52]). Here ν is the Poisson ratio with $\lambda = 2\mu\nu/(1-2\nu)$ and $\nu = \lambda/[2(\lambda + \mu)]$.

4 A non-uniformly moving dislocation loop

Investigating the non-uniform motion of a dislocation loop, we consider a closed loop of arbitrary shape (planar or non-planar) that moves arbitrary. The dislocation density tensor and the dislocation current tensor of a dislocation loop at the position $\mathbf{s}(t)$ are represented by line integrals of the form (e.g. [16, 44, 17])

$$T_{ijk}(\mathbf{r}, t) = b_i \epsilon_{jkl} \oint_{L(t)} \delta(\mathbf{r} - \mathbf{s}(t)) dL_l(\mathbf{s}(t)), \quad (33)$$

$$I_{ij}(\mathbf{r}, t) = b_i \epsilon_{jkl} \oint_{L(t)} V_k(t) \delta(\mathbf{r} - \mathbf{s}(t)) dL_l(\mathbf{s}(t)), \quad (34)$$

where $\mathbf{V} = \dot{\mathbf{s}}$ denotes the velocity of the dislocation loop at any point $\mathbf{s}(t)$ on the loop, b_i is the Burgers vector, $L(t)$ is the dislocation loop curve at time t and dL_l is a line element

³In electrodynamics, the Jefimenko formulae for the electric field strength \mathbf{E} and the magnetic field strength \mathbf{B} are given by [50, 23]:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int_V \left(\frac{\rho(\mathbf{r}', t-R/c)}{R^3} \mathbf{R} + \frac{\partial_t \rho(\mathbf{r}', t-R/c)}{cR^2} \mathbf{R} - \frac{\partial_t \mathbf{J}(\mathbf{r}', t-R/c)}{c^2 R} \right) d\mathbf{r}', \\ \mathbf{B}(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0 c^2} \int_V \left(\frac{\mathbf{J}(\mathbf{r}', t-R/c)}{R^3} + \frac{\partial_t \mathbf{J}(\mathbf{r}', t-R/c)}{cR^2} \right) \times \mathbf{R} d\mathbf{r}', \end{aligned}$$

where ρ is the electric charge density, \mathbf{J} denotes the electric current density vector, c denotes the speed of light and ϵ_0 is the permittivity of vacuum.

along the loop. Since $L(t)$ may change its position with time t , it has a more complicated structure than in the static case. $L(t)$ is the collection of all points on the dislocation line. Only subsonic source-speeds will be admitted ($|\mathbf{V}| < c_T$). If we substitute Eqs. (33) and (34) into (16) and (17) and after the integration in \mathbf{r}' , we obtain for the elastic fields of a dislocation loop

$$\begin{aligned}\beta_{im}(\mathbf{r}, t) = & \partial_k \int_{-\infty}^t \oint_{L(t')} \epsilon_{mnp} C_{jkl n} G_{ij}(\mathbf{r} - \mathbf{s}(t'), t - t') b_l dL_p(\mathbf{s}(t')) dt' \\ & + \partial_t \int_{-\infty}^t \oint_{L(t')} \rho G_{ij}(\mathbf{r} - \mathbf{s}(t'), t - t') b_j \epsilon_{mnp} V_n dL_p(\mathbf{s}(t')) dt'\end{aligned}\quad (35)$$

and

$$v_i(\mathbf{r}, t) = \partial_k \int_{-\infty}^t \oint_{L(t')} C_{jkl m} G_{ij}(\mathbf{r} - \mathbf{s}(t'), t - t') b_l \epsilon_{mnp} V_n dL_p(\mathbf{s}(t')) dt', \quad (36)$$

which are in agreement with the formulae given by Markenscoff [19].

Integration in the expressions (35) and (36) may be performed in time. In this way, we find the elastic fields as line integrals around the loop $L(t')$

$$\beta_{im}(\mathbf{r}, t) = \partial_k \oint_{L(t')} \epsilon_{mnp} C_{jkl n} \phi_{ij} b_l dL_p(\mathbf{s}(t')) + \partial_t \oint_{L(t')} \rho A_{ijk} b_j \epsilon_{mnp} dL_p(\mathbf{s}(t')) \quad (37)$$

and

$$v_i(\mathbf{r}, t) = \partial_k \oint_{L(t')} C_{jkl m} A_{ijn} b_l \epsilon_{mnp} dL_p(\mathbf{s}(t')), \quad (38)$$

where we have introduced the elastodynamic Liénard-Wiechert tensor potentials ϕ_{ij} and A_{ijk} in terms of the elastodynamic Green tensor $G_{ij}(\mathbf{r} - \mathbf{s}(t'), t - t')$:

$$\begin{aligned}\phi_{ij}(\mathbf{r}, t) = & \int_{-\infty}^t \int_{-\infty}^{\infty} G_{ij}(\mathbf{r} - \mathbf{r}', t - t') \delta(\mathbf{r}' - \mathbf{s}(t')) d\mathbf{r}' dt' \\ = & \int_{-\infty}^t G_{ij}(\mathbf{r} - \mathbf{s}(t'), t - t') dt',\end{aligned}\quad (39)$$

$$\begin{aligned}A_{ijk}(\mathbf{r}, t) = & \int_{-\infty}^t \int_{-\infty}^{\infty} G_{ij}(\mathbf{r} - \mathbf{r}', t - t') V_k(t') \delta(\mathbf{r}' - \mathbf{s}(t')) d\mathbf{r}' dt' \\ = & \int_{-\infty}^t G_{ij}(\mathbf{r} - \mathbf{s}(t'), t - t') V_k(t') dt' .\end{aligned}\quad (40)$$

More precisely, Eqs. (39) and (40) are the elastodynamical Liénard-Wiechert tensor potentials corresponding to delta-point sources acting on the position $\mathbf{s}(t')$ which have to be integrated over the loop line element $dL_p(\mathbf{s}(t'))$ in Eqs. (37) and (38). The property of the Green tensor (18) leads to the following wave equations for the elastodynamic Liénard-Wiechert tensor potentials (39) and (40)

$$[\delta_{ik} \rho \partial_{tt} - C_{ijkl} \partial_j \partial_l] \phi_{km} = \delta_{im} \delta(\mathbf{r} - \mathbf{s}(t)), \quad (41)$$

$$[\delta_{ik} \rho \partial_{tt} - C_{ijkl} \partial_j \partial_l] A_{kmn} = \delta_{im} V_n(t) \delta(\mathbf{r} - \mathbf{s}(t)), \quad (42)$$

with Eq. (19).

Substituting the elastodynamic Green tensor (21) into Eqs. (39) and (40), the integration in time may be performed. Before the integration $R = |\mathbf{r} - \mathbf{r}'|$ is a function of \mathbf{r} and \mathbf{r}' ; after the integration, which fixes $\mathbf{r}' = \mathbf{s}(t)$, $R = |\mathbf{r} - \mathbf{s}(t)|$ is a function of \mathbf{r} and $\mathbf{s}(t)$. The structure of the Green tensor (21) produces three characteristic integrals which we have to calculate. We express the integrals in terms of retarded variables by appeal to the relation [53, 54, 13]

$$\int \delta(f(t')) g(t') dt' = \frac{g(t')}{|df/dt'|} \Big|_{\text{at } f(t')=0}. \quad (43)$$

Mathematically, the factor $1/|df/dt'|$ is the Jacobian of the transformation from t' to the new integration variable $f(t')$. This mapping between the two variables is one-to-one if the Jacobian is different from zero. A sufficient condition for this is that the velocity of the source (dislocation) is less than the wave speed (subsonic motion). The first integral can be carried out with

$$\int \frac{\delta(t - t' - |\mathbf{r} - \mathbf{s}(t')|/c_T)}{|\mathbf{r} - \mathbf{s}(t')|} dt' = \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_T} \Big|_{t'=t_T}, \quad (44)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{s}(t')$ and $\mathbf{V} = \mathbf{V}(t')$. \mathbf{R} is the distance vector from the position of the source \mathbf{s} , the sender of elastic waves, to the point of the observer \mathbf{r} , the receiver of the elastic waves. The second integral is

$$\int \frac{(\mathbf{r} - \mathbf{s}(t'))_i (\mathbf{r} - \mathbf{s}(t'))_j}{|\mathbf{r} - \mathbf{s}(t')|^3} \delta(t - t' - |\mathbf{r} - \mathbf{s}(t')|/c_{L,T}) dt' = \frac{R_i R_j}{R^2} \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_{L,T}} \Big|_{t'=t_{L,T}}. \quad (45)$$

Using the relation (23), we perform the third integral as follows

$$\begin{aligned} & \int \left(\frac{3(\mathbf{r} - \mathbf{s}(t'))_i (\mathbf{r} - \mathbf{s}(t'))_j}{|\mathbf{r} - \mathbf{s}(t')|^3} - \frac{\delta_{ij}}{|\mathbf{r} - \mathbf{s}(t')|} \right) \int_{1/c_L}^{1/c_T} \kappa \delta(t - t' - \kappa |\mathbf{r} - \mathbf{s}(t')|) d\kappa dt' \\ &= \int_{1/c_L}^{1/c_T} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa d\kappa}{R - \kappa \mathbf{R} \cdot \mathbf{V}} \Big|_{t'=t_\kappa}. \end{aligned} \quad (46)$$

From the argument that the delta functions vanish $f(t') = 0$ in Eqs. (44)–(46), we obtain the condition

$$t - t' - |\mathbf{r} - \mathbf{s}(t')|/c = 0, \quad \text{with} \quad c = c_T, c_L, 1/\kappa. \quad (47)$$

Unfortunately, the retarded times $t_c = t'(\mathbf{r}, t)$ are not given directly, and they must be determined by solving Eq. (47) what can be quite tedious. Only in some simple cases t_c is easy to find. If the dislocation loop is moving with subsonic speed, the solution of Eq. (47) is unique. The retarded times are a result of the finite speeds of propagation for elastodynamic waves. In Eqs. (44)–(46) we have used the relation

$$\left| \frac{df(t')}{dt'} \right|_{t'=t_{\text{ret}}} = 1 - \frac{\mathbf{R} \cdot \mathbf{V}}{c R} \Big|_{t'=t_{\text{ret}}} > 0 \quad \text{for } |\mathbf{V}| < c_T, \quad c = c_T, c_L, 1/\kappa \quad \text{and } t_{\text{ret}} = t_T, t_L, t_\kappa, \quad (48)$$

where $f(t') = t - t' - |\mathbf{r} - \mathbf{s}(t')|/c$.

Carrying out the t' -integration in Eqs. (39) and (40), we find the explicit expressions for the elastodynamic Liénard-Wiechert tensor potentials of a ‘point dislocation source’ acting on the position $\mathbf{s}(t')$

$$4\pi\rho\phi_{ij}(\mathbf{r}, t) = \frac{1}{c_T^2} \left[\left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_T} \right] \Big|_{t'=t_T} + \frac{1}{c_L^2} \left[\frac{R_i R_j}{R^2} \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_L} \right] \Big|_{t'=t_L} \\ + \int_{1/c_L}^{1/c_T} \left[\left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa d\kappa}{R - \kappa \mathbf{R} \cdot \mathbf{V}} \right] \Big|_{t'=t_\kappa} \quad (49)$$

$$4\pi\rho A_{ijk}(\mathbf{r}, t) = \frac{1}{c_T^2} \left[\left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_T} \right] \Big|_{t'=t_T} + \frac{1}{c_L^2} \left[\frac{R_i R_j}{R^2} \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_L} \right] \Big|_{t'=t_L} \\ + \int_{1/c_L}^{1/c_T} \left[\left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{V_k \kappa d\kappa}{R - \kappa \mathbf{R} \cdot \mathbf{V}} \right] \Big|_{t'=t_\kappa}, \quad (50)$$

where \mathbf{R} and \mathbf{V} are to be evaluated at the corresponding retarded times. The elastodynamic retarded potentials (49) and (50) fulfill

$$A_{ijk}(\mathbf{r}, t) = V_k(t_{\text{ret}}) \phi_{ij}(\mathbf{r}, t). \quad (51)$$

The first δ_{ij} -terms in Eqs. (49) and (50) have an analogous form as the original Liénard-Wiechert potentials of a point charge in the electromagnetic theory⁴ (see, e.g., [22, 23]). The Liénard-Wiechert tensor potential (49) is analogous to the displacement vector, which is the Liénard-Wiechert vector potential, of a non-uniformly moving point force in elastodynamics found by Lazar [56]. Due to the appearance of two velocities of the elastic waves, the elastodynamic Liénard-Wiechert tensor potentials have a more complicated but rather straightforward structure. Like the retarded elastic fields (24) and (25), the Liénard-Wiechert tensor potentials (49) and (50) consist of three characteristic terms, again the first term corresponds to the S -wave motion, the second term represents the P -wave motion, and the third term corresponds to a combination of P and S motion. Thus, at the retarded positions the dislocation loop is the source of S -, P - and the combination of P - and S -waves.

Finally, if we substitute Eqs. (49) and (50) in Eqs. (37) and (38), the elastic fields read

⁴The original Liénard-Wiechert potentials (scalar potential ϕ and vector potential \mathbf{A}) of a point charge read [55, 50]:

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R(t') - \mathbf{R}(t') \cdot \mathbf{V}(t')/c} \right]_{t'=t_c}, \quad \mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \left[\frac{\mathbf{V}(t')}{R(t') - \mathbf{R}(t') \cdot \mathbf{V}(t')/c} \right]_{t'=t_c},$$

where q is the electric charge. Here t_c denotes the retarded time with respect to the velocity of light. They fulfill the Lorentz gauge condition: $\dot{\phi} + c^2 \text{div} \mathbf{A} = 0$.

in terms of the Liénard-Wiechert tensor potentials

$$\begin{aligned}
\beta_{im}(\mathbf{r}, t) = & \frac{1}{4\pi\rho} C_{jklm} b_l \epsilon_{mnp} \partial_k \left\{ \frac{1}{c_T^2} \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \right\} \Big|_{t'=t_T} \\
& + \frac{1}{c_L^2} \left[\oint_{L(t')} \frac{R_i R_j}{R^2} \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& + \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \Big\} \\
& + \frac{1}{4\pi} b_j \epsilon_{mkp} \partial_t \left\{ \frac{1}{c_T^2} \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \right\} \Big|_{t'=t_T} \\
& + \frac{1}{c_L^2} \left[\oint_{L(t')} \frac{R_i R_j}{R^2} \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& + \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa V_k}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \Big\} \quad (52)
\end{aligned}$$

and

$$\begin{aligned}
v_i(\mathbf{r}, t) = & \frac{1}{4\pi\rho} C_{jklm} b_l \epsilon_{mnp} \partial_k \left\{ \frac{1}{c_T^2} \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{V_n}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \right\} \Big|_{t'=t_T} \\
& + \frac{1}{c_L^2} \left[\oint_{L(t')} \frac{R_i R_j}{R^2} \frac{V_n}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& + \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa V_n}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \Big\}, \quad (53)
\end{aligned}$$

where again $\mathbf{R} = \mathbf{r} - \mathbf{s}(t')$ and $\mathbf{V} = \dot{\mathbf{s}}(t')$ are to be evaluated at the corresponding retarded times which must be determined by solving Eq. (47). The fields (52) and (53) must be evaluated at some earlier times t' (the retarded times) and for the corresponding point $\mathbf{s}(t')$ on the dislocation loop. Also the line element dL_p depends at every point $\mathbf{s}(t')$ on the retarded times. Thus, the elastic fields are given as line integrals around the dislocation line in terms of the Liénard-Wiechert tensor potentials of point sources. Due to the retarded times in Eqs. (52) and (53), $L(t')$ depends on the retarded times and three curves $L(t_T)$, $L(t_L)$ and $L(t_\kappa)$ appear in Eqs. (52) and (53). Because of the explicit dependence of the retarded times on $\mathbf{s}(t')$, every point $\mathbf{s}(t')$ on the loop $L(t')$ depends on its own retarded time. For a dislocation loop, the time dependence of the elastic fields is based on a retardation due to the retarded times which are functions of the variables \mathbf{r} , \mathbf{s} , and t . The elastic fields (52) and (53) at the point \mathbf{r} and at time t receive contribution from every point $\mathbf{s}(t_{\text{ret}})$ on the moving loop sending the elastic waves (S, P , mixed waves) at the retarded times (t_T, t_L, t_κ) . For a moving loop these times are different not only due to different points on the loop, but also because the loop moves. The evaluation of the fields (52) and (53) is far from a trivial task. Thus, Eqs. (52) and (53) are complicated line integrals depending on the retarded times, and time and spatial derivatives outside the line integrals.

Alternatively, the substitution of Eqs. (33) and (34) into the retarded elastic fields (24) and (25) gives directly the fields (52) and (53). Thus, the tensor structure of (52) and (53)

is inherited from the tensor structure of the retarded elastic fields (24) and (25). Because the time-variable of the sources \mathbf{T} and \mathbf{I} in Eqs. (24) and (25) is the retarded time t_{ret} , consequently the closed loop curve depends on the retarded times $L(t_{\text{ret}})$ in Eqs. (52) and (53). Although the structure of Eqs. (52) and (53) is elegant from the mathematical point of view, the evaluation of such expressions for a non-uniformly moving dislocation loop is very complicated due to the dependence of the retarded times. But this is the price we have to pay if we perform the integration in time and use the three-dimensional Green tensor (21) in order to obtain the elastodynamic Liénard-Wiechert tensor potentials. Since the relation of the retarded position to the present position of the dislocation loop is not in general, known, the elastic fields and the Liénard-Wiechert tensor potentials ordinarily permit only the evaluation of the fields in terms of the retarded positions and velocities of the dislocation loop. The complexity of the fields (52) and (53) is hidden behind the elegance of these formulae.

On the other hand, if we substitute Eqs. (33) and (34) into the Jefimenko type formulae (30) and (31) and carry out the integration in \mathbf{r}' , we obtain

$$\begin{aligned}
\beta_{im}(\mathbf{r}, t) = & -\frac{1}{4\pi\rho} C_{jklm} b_l \epsilon_{mnp} \left\{ \frac{1}{c_T^2} \left[\oint_{L(t')} \left(\frac{\delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j}{R^2} \right. \right. \right. \\
& \left. \left. \left. - \frac{3R_i R_j R_k}{R^4} \right) \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \right|_{t'=t_T} \\
& + \frac{1}{c_T^3} \partial_t \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{R_k}{R} \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_T} \\
& - \frac{1}{c_L^2} \left[\oint_{L(t')} \left(\frac{\delta_{jk} R_i + \delta_{ik} R_j}{R^2} - \frac{3R_i R_j R_k}{R^4} \right) \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& + \frac{1}{c_L^3} \partial_t \left[\oint_{L(t')} \frac{R_i R_j R_k}{R^3} \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& - \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{\delta_{ij} R_k + 3\delta_{jk} R_i + 3\delta_{ik} R_j}{R^2} - \frac{9R_i R_j R_k}{R^4} \right) \frac{\kappa}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \\
& + \partial_t \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{R_k}{R} \frac{\kappa^2}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \Big\} \\
& + \frac{1}{4\pi} b_j \epsilon_{mkp} \left\{ \frac{1}{c_T^2} \partial_t \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \right|_{t'=t_T} \\
& + \frac{1}{c_L^2} \partial_t \left[\oint_{L(t')} \frac{R_i R_j}{R^2} \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& + \partial_t \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa V_k}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \Big\} \quad (54)
\end{aligned}$$

and

$$\begin{aligned}
v_i(\mathbf{r}, t) = & -\frac{1}{4\pi\rho} C_{jklm} b_l \epsilon_{mnp} \left\{ \frac{1}{c_T^2} \left[\oint_{L(t')} \left(\frac{\delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j}{R^2} \right. \right. \right. \\
& \left. \left. \left. - \frac{3R_i R_j R_k}{R^4} \right) \frac{V_n}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \right|_{t'=t_T} \\
& + \frac{1}{c_T^3} \partial_t \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{R_k}{R} \frac{V_n}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_T} \\
& - \frac{1}{c_L^2} \left[\oint_{L(t')} \left(\frac{\delta_{jk} R_i + \delta_{ik} R_j}{R^2} - \frac{3R_i R_j R_k}{R^4} \right) \frac{V_n}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& + \frac{1}{c_L^3} \partial_t \left[\oint_{L(t')} \frac{R_i R_j R_k}{R^3} \frac{V_n}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_L} \\
& - \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{\delta_{ij} R_k + 3\delta_{jk} R_i + 3\delta_{ik} R_j}{R^2} - \frac{9R_i R_j R_k}{R^4} \right) \frac{\kappa V_n}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \\
& + \partial_t \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{R_k}{R} \frac{\kappa^2 V_n}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_p(\mathbf{s}(t')) \right] \Big|_{t'=t_\kappa} d\kappa \Big\}. \tag{55}
\end{aligned}$$

Eqs. (54) and (55) are analogous to the Heaviside-Feynman formulae [57, 58] of electromagnetic theory and we may call them the Heaviside-Feynman type formulae for a dislocation loop. They are more complicated than the original Heaviside-Feynman formulae [57, 58] (see also [59, 23, 51, 60]) for a point charge in electrodynamics⁵. Eqs. (54) and (55) contain terms without derivatives and terms with time-derivatives. They still possess a quite complicated tensor structure and line integrals depending on the retarded times. The causal dependencies of the elastic fields of a non-uniformly moving dislocation loop are described by retarded line integrals along the loop.

5 Static limit of a dislocation loop

In this section, we give the static limit of the Liénard-Wiechert tensor potentials and of the elastic distortion of a dislocation loop as a check of the above results.

In order to carry out the static limit, we set $\mathbf{V} = 0$ and $\mathbf{s}(t') = \mathbf{r}'$ and we substitute Eq. (22) and $\lambda = 2\mu\nu/(1 - 2\nu)$ into Eq. (49). If we perform the integration in κ and arrange in proper order the appearing terms, we find

$$\phi_{ij} = G_{ij} \tag{56}$$

⁵In electrodynamics, the Heaviside-Feynman formulae for the electric field strength \mathbf{E} and the magnetic field strength \mathbf{B} of a non-uniformly moving point charge are of the form [51, 23]:

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) = & \frac{q}{4\pi\epsilon_0} \left(\left[\frac{\mathbf{R}}{R^2(R - \mathbf{R} \cdot \mathbf{V}/c)} \right]_{t_c} + \frac{1}{c} \partial_t \left[\frac{\mathbf{R}}{R(R - \mathbf{R} \cdot \mathbf{V}/c)} \right]_{t_c} - \frac{1}{c^2} \partial_t \left[\frac{\mathbf{V}}{R - \mathbf{R} \cdot \mathbf{V}/c} \right]_{t_c} \right), \\
\mathbf{B}(\mathbf{r}, t) = & \frac{q}{4\pi\epsilon_0 c^2} \left(\left[\frac{\mathbf{V} \times \mathbf{R}}{R^2(R - \mathbf{R} \cdot \mathbf{V}/c)} \right]_{t_c} + \frac{1}{c} \partial_t \left[\frac{\mathbf{V} \times \mathbf{R}}{R(R - \mathbf{R} \cdot \mathbf{V}/c)} \right]_{t_c} \right).
\end{aligned}$$

with

$$G_{ij} = \frac{1}{16\pi\mu(1-\nu)} [2(1-\nu)\delta_{ij}\Delta - \partial_i\partial_j] R, \quad (57)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Eq. (57) is the static three-dimensional Green tensor of the Navier equation [61, 1]. That means that the Liénard-Wiechert tensor potential ϕ_{ij} reduces to the static elastic Green tensor in the static limit. In this manner, we find from Eq. (37) the so-called Mura formula (e.g. [15, 43, 62]) as static limit

$$\beta_{im}(\mathbf{r}) = \oint_L \epsilon_{mnp} C_{jkl n} G_{ij,k} b_l dL'_p, \quad (58)$$

which gives the elastic distortion of a static dislocation loop.

On the other hand, the static limit of Eq. (54) is obtained as

$$\beta_{im}(\mathbf{r}) = -\frac{b_l}{8\pi(1-\nu)} \oint_L \epsilon_{mnp} \left\{ (1-2\nu) \frac{\delta_{il}R_n + \delta_{in}R_l - \delta_{ln}R_i}{R^3} + \frac{3R_iR_lR_n}{R^5} \right\} dL'_p, \quad (59)$$

which is the explicit form of Eq. (58).

6 Straight dislocations

We derive now the two-dimensional Liénard-Wiechert tensor potentials and the elastic fields of straight dislocations in our general framework. We choose the dislocation line ℓ_z parallel to the z -axis. The dislocation density and dislocation current tensors of edge dislocations are given by

$$T_{ijk} = b_i \epsilon_{jkz} \ell_z \delta(\mathbf{R}), \quad I_{ij} = b_i \epsilon_{jkz} V_k \ell_z \delta(\mathbf{R}). \quad (60)$$

For a screw dislocation, the dislocation density and dislocation current tensors read

$$T_{zjk} = b_z \epsilon_{jkz} \ell_z \delta(\mathbf{R}), \quad I_{zj} = b_z \epsilon_{jkz} V_k \ell_z \delta(\mathbf{R}), \quad (61)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{s}(t) \in \mathbb{R}^2$ and $i, j, k = x, y$. The tensors (60) and (61) fulfill, respectively, the relations

$$I_{ij} = V_k T_{ijk}, \quad I_{zj} = V_k T_{zjk}. \quad (62)$$

Eq. (62) gives a relation between the dislocation current tensor, the dislocation density tensor and the dislocation velocity vector, which is valid for a single straight dislocation. Günther [39] and Teodosiu [36] derived such a relation between \mathbf{I} and \mathbf{T} for uniformly moving dislocations, that means for constant dislocation velocity \mathbf{V} . If the relation (62) is valid, then \mathbf{I} is a convection dislocation current.

If the material is infinitely extended, the two-dimensional elastodynamic Green tensor of plane-strain reads [48, 63]

$$G_{ik}(\mathbf{r}, t) = \frac{1}{2\pi\rho} \left\{ \frac{x_i x_k}{r^4} \left(\frac{[2t^2 - r^2/c_L^2]}{\sqrt{t^2 - r^2/c_L^2}} H(t - r/c_L) - \frac{[2t^2 - r^2/c_T^2]}{\sqrt{t^2 - r^2/c_T^2}} H(t - r/c_T) \right) - \frac{\delta_{ik}}{r^2} \left(\sqrt{t^2 - r^2/c_L^2} H(t - r/c_L) - \frac{t^2}{\sqrt{t^2 - r^2/c_T^2}} H(t - r/c_T) \right) \right\} \quad (63)$$

and the elastodynamic Green tensor of anti-plane strain is given by

$$G_{zz}(\mathbf{r}, t) = \frac{1}{2\pi\rho c_T^2} \frac{H(t - r/c_T)}{\sqrt{t^2 - r^2/c_T^2}}, \quad (64)$$

where $H(\cdot)$ denotes the Heaviside step function and $r = \sqrt{x^2 + y^2}$.

If we substitute Eqs. (63) and (60) in Eqs. (16) and (17) and perform the integration in \mathbf{r}' , we find for the elastic fields (no summation over z in Eqs. (65) and (66))

$$\beta_{im}(\mathbf{r}, t) = \epsilon_{mnz} C_{jklm} \partial_k \phi_{ij} b_l \ell_z + \rho \partial_t A_{ijk} b_j \epsilon_{mkz} \ell_z \quad (65)$$

and

$$v_i(\mathbf{r}, t) = C_{jklm} \partial_k A_{ijn} b_l \epsilon_{mnz} \ell_z. \quad (66)$$

Using the definitions of the elastodynamic Liénard-Wiechert tensor potentials (39) and (40) and the Green tensor (63), we have here introduced the two-dimensional Liénard-Wiechert tensor potentials

$$\begin{aligned} \phi_{ij}(\mathbf{r}, t) = \frac{1}{2\pi\rho} & \left[\int_{-\infty}^{t_L} \left(\frac{R_i R_j}{R^4} \frac{\bar{t}^2}{S_L} + \left(\frac{R_i R_j - \delta_{ij} R^2}{R^4} \right) S_L \right) dt' \right. \\ & \left. - \int_{-\infty}^{t_T} \left(\frac{R_i R_j}{R^4} S_T + \left(\frac{R_i R_j - \delta_{ij} R^2}{R^4} \right) \frac{\bar{t}^2}{S_T} \right) dt' \right], \end{aligned} \quad (67)$$

$$\begin{aligned} A_{ijk}(\mathbf{r}, t) = \frac{1}{2\pi\rho} & \left[\int_{-\infty}^{t_L} V_k(t') \left(\frac{R_i R_j}{R^4} \frac{\bar{t}^2}{S_L} + \left(\frac{R_i R_j - \delta_{ij} R^2}{R^4} \right) S_L \right) dt' \right. \\ & \left. - \int_{-\infty}^{t_T} V_k(t') \left(\frac{R_i R_j}{R^4} S_T + \left(\frac{R_i R_j - \delta_{ij} R^2}{R^4} \right) \frac{\bar{t}^2}{S_T} \right) dt' \right]. \end{aligned} \quad (68)$$

The notation here is

$$R_i = x_i - s_i(t'), \quad \bar{t} = t - t', \quad S_T^2 = \bar{t}^2 - \frac{R^2}{c_T^2}, \quad S_L^2 = \bar{t}^2 - \frac{R^2}{c_L^2}. \quad (69)$$

The two retarded times $t_T = t'$ and $t_L = t'$ are the roots of $S_T^2 = 0$ and $S_L^2 = 0$, respectively, which are less than t . The solving of the conditions $S_T^2 = 0$ and $S_L^2 = 0$ is for a general motion non-trivial and can be very complicated. For subsonic dislocations the solutions for the retarded times t_T and t_L are unique. The general expressions (65)–(68) contain the fields for gliding as well as climbing dislocations. If $\mathbf{V} \parallel \mathbf{b}$, the expressions (65)–(68) describe a gliding edge dislocation and if $\mathbf{V} \perp \mathbf{b}$, Eqs. (65)–(68) give the fields of a climbing edge dislocation (see, e.g., [8]). For these cases, we can recover from Eqs. (65)–(68) the explicit expressions given by Kiusalaas and Mura [6], Lardner [1] and Lazar [8].

For a screw dislocation, if we insert Eqs. (64) and (61) in Eqs. (16) and (17), and performing the integration in \mathbf{r}' , we obtain for the elastic fields

$$\beta_{zm}(\mathbf{r}, t) = \epsilon_{mnz} C_{zkzn} \partial_k \phi_{zz} b_z \ell_z + \rho \partial_t A_{zzk} b_z \epsilon_{mkz} \ell_z \quad (70)$$

and

$$v_z(\mathbf{r}, t) = C_{zkzm} \partial_k A_{zzn} b_z \epsilon_{mnz} \ell_z. \quad (71)$$

Using the definitions Eqs. (49) and (50) and the Green function (64), the two-dimensional Liénard-Wiechert potentials of anti-plane strain are

$$\phi_{zz}(\mathbf{r}, t) = \frac{1}{2\pi\rho c_T^2} \int_{-\infty}^{t_T} \frac{1}{S_T} dt', \quad A_{zzk}(\mathbf{r}, t) = \frac{1}{2\pi\rho c_T^2} \int_{-\infty}^{t_T} \frac{V_k(t')}{S_T} dt'. \quad (72)$$

It should be noted that the index z in Eqs. (70)–(72) is a fixed index (no summation over z) due to the reduction from 3D to the anti-plane strain problem. From the general expressions (70)–(72), we can reduce all the field components given by Eshelby [5], Nabarro [64], Lardner [1], and Lazar [8].

The two-dimensional Liénard-Wiechert potentials (67), (68), and (72) are time-integrals over the history of the motion and thus, they possess an afterglow. For that reason, a straight dislocation is haunted by its past as Eshelby [4] mentioned. Nevertheless, for the two-dimensional Liénard-Wiechert potentials (67), (68) and (72) we are left to evaluate time-integrals of considerable complexity which only in some simple cases yield results of elementary functions in a closed form. Also the calculation of the retarded times is not a trivial task. For straight dislocations, the static limit of the elastic fields of non-uniformly moving straight dislocations was given in [8].

7 Near-field approximation of straight dislocations

In the near-field approximation, it is important to determine the character of the singularities of Eqs. (67), (68) and (72) at $\mathbf{R} \rightarrow 0$. Thus, we have to calculate only the near-field approximation of the integrals in Eqs. (67), (68) and (72). We follow the near-field approximation for cylindrical waves given by Courant and Hilbert [65], Whitham [66] and Barton [13]. We consider here dislocations which are at rest and at $t' = 0$ they start to move. This problem can be viewed as the superposition of the static equilibrium problem for $t < 0$ with the dynamic problem for $t > 0$. Since the wave propagation nature of the solution is of interest, we consider the superposition-related problem obtained by subtracting the equilibrium solution from the complete solution. Therefore, the lower integration limits of the Liénard-Wiechert potentials (67), (68) and (72) are changed from $-\infty$ to 0 (see also [67, 68, 69, 21]).

We start with the near-field approximation for a screw dislocation. In the near-field approximation, the retardation is negligible. Thus, we have $\mathbf{V}(t') \approx \mathbf{V}(t)$ and $\mathbf{R} =$

$\mathbf{r} - \mathbf{s}(t') \approx \mathbf{r} - \mathbf{s}(t)$. The two-dimensional Liénard-Wiechert potentials (72) reduce to

$$\begin{aligned}
2\pi\rho c_T^2 \phi_{zz}(\mathbf{r}, t) &\approx \int_0^t \frac{H(\tau - R/c_T)}{\sqrt{\tau^2 - R^2/c_T^2}} d\tau', \quad \tau = t - t' \\
&= \int_0^t \frac{H(\tau - R/c_T)}{\sqrt{\tau^2 - R^2/c_T^2}} d\tau \\
&= H(t - R/c_T) \int_{R/c_T}^t \frac{d\tau}{\sqrt{\tau^2 - R^2/c_T^2}} \\
&= H(t - R/c_T) \ln \left[\frac{c_T t}{R} + \sqrt{\frac{c_T^2 t^2}{R^2} - 1} \right] \\
&\approx H(t - R/c_T) \ln \frac{2c_T t}{R} + \mathcal{O}(R^2)
\end{aligned} \tag{73}$$

and $A_{zzk}(\mathbf{r}, t) = V_k(t)\phi_{zz}(\mathbf{r}, t)$. For $t \gg R/c_T$, Eq. (73) simplifies to (see also [13])

$$\phi_{zz}(\mathbf{r}, t) \approx \frac{1}{2\pi\rho c_T^2} \ln \frac{2c_T t}{R}, \quad A_{zzk}(\mathbf{r}, t) \approx \frac{1}{2\pi\rho c_T^2} V_k(t) \ln \frac{2c_T t}{R}. \tag{74}$$

If we substitute (74) into Eqs. (70) and (71), we obtain the singular terms of the elastic fields of a moving screw dislocation

$$\beta_{zm}(\mathbf{r}, t) = -\epsilon_{mnz} \frac{b_z \ell_z}{2\pi\rho c_T^2} \left(C_{zkzn} \frac{R_k}{R^2} - \rho V_n(t) \frac{\mathbf{R} \cdot \mathbf{V}(t)}{R^2} - \rho \dot{V}_n(t) \ln \frac{2c_T t}{R} \right) \tag{75}$$

and

$$v_z(\mathbf{r}, t) = -\epsilon_{mnz} \frac{b_z \ell_z}{2\pi\mu} C_{zkzm} \frac{R_k V_n(t)}{R^2}. \tag{76}$$

Explicitly, they read

$$\beta_{zx}(\mathbf{r}, t) = -\frac{b_z \ell_z}{2\pi} \left\{ \left(1 - \frac{V_y^2(t)}{c_T^2} \right) \frac{R_y}{R^2} - \frac{V_x(t)V_y(t)}{c_T^2} \frac{R_x}{R^2} - \frac{\dot{V}_y(t)}{c_T^2} \ln \frac{2c_T t}{R} \right\} \tag{77}$$

$$\beta_{zy}(\mathbf{r}, t) = \frac{b_z \ell_z}{2\pi} \left\{ \left(1 - \frac{V_x^2(t)}{c_T^2} \right) \frac{R_x}{R^2} - \frac{V_x(t)V_y(t)}{c_T^2} \frac{R_y}{R^2} - \frac{\dot{V}_x(t)}{c_T^2} \ln \frac{2c_T t}{R} \right\} \tag{78}$$

$$v_z(\mathbf{r}, t) = \frac{b_z \ell_z}{2\pi} \frac{R_y V_x(t) - R_x V_y(t)}{R^2}. \tag{79}$$

It is obvious in Eqs. (77)–(79) that a screw dislocation contains a $1/R$ singularity and a logarithmic singularity in the near-field. The acceleration terms give the logarithmic singularity.

Also Eqs. (77) and (78) contain the correct static limit with $\mathbf{V} = 0$ and $\mathbf{s}(t) = \text{constant}$ for the elastic distortion of a screw dislocation with the Burgers vector $\mathbf{b} = (0, 0, b_z)$ given by deWit [70]

$$\beta_{zx} = -\frac{b_z}{2\pi} \frac{y}{r^2}, \tag{80}$$

$$\beta_{zy} = \frac{b_z}{2\pi} \frac{x}{r^2}. \tag{81}$$

For convenience we have chosen $\mathbf{s} = 0$.

For the near-fields of edge dislocations, we need the following integral approximations

$$\begin{aligned}
\int_0^t \frac{H(\tau - R/c) \tau^2}{\sqrt{\tau^2 - R^2/c^2}} dt' &= \int_0^t \frac{H(\tau - R/c) \tau^2}{\sqrt{\tau^2 - R^2/c^2}} d\tau \\
&= H(t - R/c) \int_{R/c}^t \frac{\tau^2 d\tau}{\sqrt{\tau^2 - R^2/c^2}} \\
&= H(t - R/c) \left(\frac{R^2}{2c^2} \ln \left[\frac{ct}{R} + \sqrt{\frac{c^2 t^2}{R^2} - 1} \right] + \frac{t}{2} \sqrt{t^2 - \frac{R^2}{c^2}} \right) \\
&\approx H(t - R/c) \left(\frac{R^2}{2c^2} \ln \frac{2ct}{R} + \frac{t^2}{2} - \frac{R^2}{4c^2} + \mathcal{O}(R^4) \right) \tag{82}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^t H(\tau - R/c) \sqrt{\tau^2 - R^2/c^2} dt' &= \int_0^t H(\tau - R/c) \sqrt{\tau^2 - R^2/c^2} d\tau \\
&= H(t - R/c) \int_{R/c}^t \sqrt{\tau^2 - R^2/c^2} d\tau \\
&= H(t - R/c) \left(-\frac{R^2}{2c^2} \ln \left[\frac{ct}{R} + \sqrt{\frac{c^2 t^2}{R^2} - 1} \right] + \frac{t}{2} \sqrt{t^2 - \frac{R^2}{c^2}} \right) \\
&\approx H(t - R/c) \left(-\frac{R^2}{2c^2} \ln \frac{2ct}{R} + \frac{t^2}{2} - \frac{R^2}{4c^2} + \mathcal{O}(R^4) \right). \tag{83}
\end{aligned}$$

Using Eqs. (82) and (83), we obtain for the Liénard-Wiechert tensor potentials (67) and (68)

$$\begin{aligned}
2\pi\rho\phi_{ij}(\mathbf{r}, t) &\approx H(t - R/c_L) \left\{ \frac{R_i R_j}{R^4} t \sqrt{t^2 - \frac{R^2}{c_L^2}} \right. \\
&\quad \left. + \delta_{ij} \left(\frac{1}{2c_L^2} \ln \left[\frac{c_L t}{R} + \sqrt{\frac{c_L^2 t^2}{R^2} - 1} \right] - \frac{t}{2R^2} \sqrt{t^2 - \frac{R^2}{c_L^2}} \right) \right\} \\
&\quad - H(t - R/c_T) \left\{ \frac{R_i R_j}{R^4} t \sqrt{t^2 - \frac{R^2}{c_T^2}} \right. \\
&\quad \left. - \delta_{ij} \left(\frac{1}{2c_T^2} \ln \left[\frac{c_T t}{R} + \sqrt{\frac{c_T^2 t^2}{R^2} - 1} \right] + \frac{t}{2R^2} \sqrt{t^2 - \frac{R^2}{c_T^2}} \right) \right\} \tag{84}
\end{aligned}$$

and $A_{ijk}(\mathbf{r}, t) = V_k(t) \phi_{ij}(\mathbf{r}, t)$. If $t \gg R/c_T$ and $t \gg R/c_L$, then (84) simplifies to

$$\phi_{ij}(\mathbf{r}, t) \approx \frac{1}{4\pi\rho} \left[\left(\frac{1}{c_T^2} - \frac{1}{c_L^2} \right) \frac{R_i R_j}{R^2} + \delta_{ij} \left(\frac{1}{c_T^2} \ln \frac{2c_T t}{R} + \frac{1}{c_L^2} \ln \frac{2c_L t}{R} \right) \right] \tag{85}$$

and $A_{ijk}(\mathbf{r}, t) = V_k(t) \phi_{ij}(\mathbf{r}, t)$. We calculate

$$\partial_k \phi_{ij}(\mathbf{r}, t) = -\frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu) \delta_{ij} \frac{R_k}{R^2} - \delta_{ik} \frac{R_j}{R^2} - \delta_{jk} \frac{R_i}{R^2} + 2 \frac{R_i R_j R_k}{R^4} \right], \quad (86)$$

$\partial_k A_{ijk}(\mathbf{r}, t) = V_k(t) \partial_k \phi_{ij}(\mathbf{r}, t)$ and

$$\begin{aligned} \partial_t A_{ijk}(\mathbf{r}, t) &= \frac{\dot{V}_k(t)}{8\pi\mu(1-\nu)} \frac{R_i R_j}{R^2} + \frac{\dot{V}_k(t) \delta_{ij}}{4\pi\rho} \left[\frac{1}{c_T^2} \ln \frac{2c_T t}{R} + \frac{1}{c_L^2} \ln \frac{2c_L t}{R} \right] \\ &+ \frac{V_k(t)}{8\pi\mu(1-\nu)} \left[(3-4\nu) \delta_{ij} \frac{\mathbf{R} \cdot \mathbf{V}}{R^2} - \frac{R_i V_j + R_j V_i}{R^2} + 2 \frac{R_i R_j \mathbf{R} \cdot \mathbf{V}}{R^4} \right]. \end{aligned} \quad (87)$$

The first term in Eq. (87) is non-singular and, thus, it does not contribute to the singular near-field. Substituting (86) and (87) into Eqs. (65) and (66), we find for the singular near-fields of the elastic fields of edge dislocations

$$\begin{aligned} \beta_{im}(\mathbf{r}, t) &= -\frac{\epsilon_{mnz} b_l \ell_z}{8\pi\mu(1-\nu)} \left(C_{jklm} \left[(3-4\nu) \delta_{ij} \frac{R_k}{R^2} - \delta_{ik} \frac{R_j}{R^2} - \delta_{jk} \frac{R_i}{R^2} + 2 \frac{R_i R_j R_k}{R^4} \right] \right. \\ &\quad \left. - \rho V_n(t) \left[(3-4\nu) \delta_{il} \frac{\mathbf{R} \cdot \mathbf{V}}{R^2} - \frac{R_i V_l + R_l V_i}{R^2} + 2 \frac{R_i R_l \mathbf{R} \cdot \mathbf{V}}{R^4} \right] \right) \\ &\quad + \frac{\epsilon_{mnz} b_l \ell_z \dot{V}_n(t)}{4\pi} \left[\frac{1}{c_T^2} \ln \frac{2c_T t}{R} + \frac{1}{c_L^2} \ln \frac{2c_L t}{R} \right] \end{aligned} \quad (88)$$

and

$$v_i(\mathbf{r}, t) = -\frac{\epsilon_{mnz} b_l \ell_z C_{jklm} V_n(t)}{8\pi\mu(1-\nu)} \left((3-4\nu) \delta_{ij} \frac{R_k}{R^2} - \delta_{ik} \frac{R_j}{R^2} - \delta_{jk} \frac{R_i}{R^2} + 2 \frac{R_i R_j R_k}{R^4} \right), \quad (89)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{s}(t)$. It can be seen in Eqs. (88) and (89) that edge dislocations contain a $1/R$ singularity and a logarithmic singularity in the near-field approximation. The acceleration terms in Eq. (88) give the logarithmic singularity. Eqs. (88) and (89) are valid for climbing and gliding edge dislocations.

It is also important to note that Eq. (88) gives the correct static limit with $\mathbf{V} = 0$ and $\mathbf{s}(t) = \text{constant}$ for the elastic distortion of an edge dislocation with Burgers vector $\mathbf{b} = (b_x, 0, 0)$ given by deWit [70]

$$\beta_{xx} = -\frac{b_x}{4\pi(1-\nu)} \frac{y}{r^2} \left[(1-2\nu) + 2 \frac{x^2}{r^2} \right], \quad (90)$$

$$\beta_{yx} = -\frac{b_x}{4\pi(1-\nu)} \frac{x}{r^2} \left[(1-2\nu) + 2 \frac{y^2}{r^2} \right], \quad (91)$$

$$\beta_{xy} = \frac{b_x}{4\pi(1-\nu)} \frac{x}{r^2} \left[(3-2\nu) - 2 \frac{y^2}{r^2} \right], \quad (92)$$

$$\beta_{yy} = -\frac{b_x}{4\pi(1-\nu)} \frac{y}{r^2} \left[(1-2\nu) - 2 \frac{x^2}{r^2} \right]. \quad (93)$$

Again we have chosen $\mathbf{s} = 0$.

It is important to mention that the history of the motion is not visible in the singularities of the non-uniformly moving dislocations (75), (76), (88), and (89). The singular terms are determined by the current situation. Both screw and edge dislocations are characterized by two types of singularities, namely, $1/R$ and logarithmic singularities. The logarithmic singularity is characteristic for the acceleration motion. The logarithmic singularities (acceleration terms) appear only in the elastic distortions (75) and (88) and not in the elastic velocity terms (76) and (89). The character of the singularities of the near fields is in agreement with the results given by Markenscoff [71], Callias *et al.* [72] and Ni and Markenscoff [73, 74]. A near-field approximation of singular integrals giving simple coefficients of the singular terms has been used. In this simple near-field approximation, we have neglected the retardation.

8 The Mura dislocation tensor potentials, the retarded dislocation tensor potentials and the Liénard-Wiechert tensor potentials of a dislocation loop

On the other hand, Mura [75, 15] has written the elastic fields in terms of so-called dislocation tensor potentials in the following way

$$\beta_{im}(\mathbf{r}, t) = \epsilon_{mnp} (C_{jkl n} \partial_k \phi_{ijlp} + \rho \partial_t A_{ijnjp}) , \quad (94)$$

$$v_i(\mathbf{r}, t) = \epsilon_{mnp} C_{jkl m} \partial_k A_{ijnlp} . \quad (95)$$

The dislocation tensor potentials ϕ_{ijkl} and A_{ijkmn} , introduced by Mura [75, 15], are defined by (see also [76])

$$\phi_{ijkl}(\mathbf{r}, t) = \int_{-\infty}^t \int_{-\infty}^{\infty} G_{ij}(\mathbf{r} - \mathbf{r}', t - t') \alpha_{kl}(\mathbf{r}', t') d\mathbf{r}' dt' , \quad (96)$$

$$A_{ijkmn}(\mathbf{r}, t) = \int_{-\infty}^t \int_{-\infty}^{\infty} G_{ij}(\mathbf{r} - \mathbf{r}', t - t') V_{kmn}(\mathbf{r}', t') d\mathbf{r}' dt' , \quad (97)$$

and they fulfill the following wave equations

$$[\delta_{ik} \rho \partial_{tt} - C_{ijkl} \partial_j \partial_l] \phi_{kmpq} = \delta_{im} \alpha_{pq} , \quad (98)$$

$$[\delta_{ik} \rho \partial_{tt} - C_{ijkl} \partial_j \partial_l] A_{kmnpq} = \delta_{im} V_{npq} , \quad (99)$$

where V_{kmn} is Mura's dislocation velocity tensor [14, 77]. Although V_{kmn} possesses 27 components, only 9 components which are skew-symmetric in some indices ($V_{kmn} = -V_{nmk}$ and $A_{ijkmn} = -A_{ijnmk}$) enter Eqs. (94) and (95). For a straight dislocation with Burgers vector b_m , dislocation line direction ℓ_n and dislocation velocity V_k , it reads: $V_{kmn} = V_k b_m \ell_n \delta(\mathbf{R}) = V_k \alpha_{mn}$. Here α_{ij} denotes the usual dislocation density tensor (10). The dislocation current tensor (7) can be written in terms of Mura's dislocation velocity tensor as follows

$$I_{ij} = \epsilon_{jkl} V_{kil} . \quad (100)$$

Substituting Eqs. (10) and (100) into Eq. (9), the Bianchi identity (9) reads

$$\dot{\alpha}_{il} + V_{kil,k} - V_{lik,k} = 0, \quad (101)$$

which leads to the following 'gauge' or side condition for the dislocation tensor potentials (see also [15])

$$\dot{\phi}_{ijkl} + A_{ijmkl,m} - A_{ijlkm,m} = 0. \quad (102)$$

If we substitute the Green tensor (21) into Eqs. (96) and (97) and perform the integration in time t' , we find the retarded dislocation tensor potentials⁶

$$\begin{aligned} \phi_{ijkl}(\mathbf{r}, t) = \frac{1}{4\pi\rho} \int_{\mathcal{V}} \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) \alpha_{kl}(\mathbf{r}', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} \alpha_{kl}(\mathbf{r}', t_L) \right. \\ \left. + \left(\frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa \alpha_{kl}(\mathbf{r}', t_\kappa) d\kappa \right\} d\mathbf{r}' \end{aligned} \quad (103)$$

and

$$\begin{aligned} A_{ijkmn}(\mathbf{r}, t) = \frac{1}{4\pi\rho} \int_{\mathcal{V}} \left\{ \frac{1}{c_T^2} \left(\frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) V_{kmn}(\mathbf{r}', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} V_{kmn}(\mathbf{r}', t_L) \right. \\ \left. + \left(\frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa V_{kmn}(\mathbf{r}', t_\kappa) d\kappa \right\} d\mathbf{r}', \end{aligned} \quad (104)$$

where the retarded times t_T , t_L and t_κ are given by Eqs. (26)–(28). Because the integrands are evaluated at the retarded times, these fields are called retarded dislocation tensor potentials. They are the causal solutions of the inhomogeneous Navier equations (98) and (99). The retarded dislocation tensor potentials (103) and (104) satisfy the gauge condition (102), this can be easily checked. In the static case, the retarded dislocation tensor potentials (103) and (104) reduce to

$$\phi_{ijkl}(\mathbf{r}) = \frac{1}{16\pi\mu(1-\nu)} \int_{\mathcal{V}} \frac{1}{R} \left[(3-4\nu)\delta_{ij} + \frac{R_i R_j}{R^2} \right] \alpha_{kl}(\mathbf{r}') d\mathbf{r}', \quad (105)$$

and $A_{ijkmn}(\mathbf{r}) = 0$. On the other hand, if we substitute Eqs. (103) and (104) into (94) and (95) and use the relations (100), (11), and (25), we obtain the Jefimenko type equations (30) and (31).

If we compare Eqs. (94) and (95) with (37) and (38), we find

$$\phi_{ijmn}(\mathbf{r}, t) = \oint_{L(t')} \phi_{ij} b_m dL_n(\mathbf{s}(t')), \quad A_{ijkmn}(\mathbf{r}, t) = \oint_{L(t')} A_{ijk} b_m dL_n(\mathbf{s}(t')), \quad (106)$$

⁶In electrodynamics, the retarded electromagnetic potentials were originally introduced by Lorenz [78] and they read [50, 23]:

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho(\mathbf{r}', t - R/c)}{R} d\mathbf{r}', \quad \mathbf{A}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 c^2} \int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}', t - R/c)}{R} d\mathbf{r}',$$

where ρ is the electric charge density and \mathbf{J} denotes the electric current density vector. The idea of a retarded scalar potential was first developed by Lorenz [79] in 1861 when studying waves in the theory of elasticity. The retarded potentials fulfill the Lorentz gauge condition: $\dot{\phi} + c^2 \text{div} \mathbf{A} = 0$ (see, e.g., [50, 49]).

that means that the dislocation tensor potentials are given as line integrals of the three-dimensional elastodynamic Liénard-Wiechert tensor potentials of a so-called ‘point dislocation source’ (49) and (50) along the dislocation line $L(t')$. The dislocation tensor potentials of a dislocation loop (106) may be identified with the actual Liénard-Wiechert tensor potentials of a dislocation loop. If we substitute Eqs. (49) and (50) into (106), they read explicitly

$$\begin{aligned} \phi_{ijmn}(\mathbf{r}, t) = \frac{b_m}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_n(\mathbf{s}(t')) \right] \right|_{t'=t_T} \\ + \frac{1}{c_L^2} \left[\oint_{L(t')} \frac{R_i R_j}{R^2} \frac{1}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_n(\mathbf{s}(t')) \right] \right|_{t'=t_L} \\ + \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_n(\mathbf{s}(t')) \right] \right|_{t'=t_\kappa} d\kappa \Big\} \end{aligned} \quad (107)$$

and

$$\begin{aligned} A_{ijkmn}(\mathbf{r}, t) = \frac{b_m}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[\oint_{L(t')} \left(\delta_{ij} - \frac{R_i R_j}{R^2} \right) \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_T} dL_n(\mathbf{s}(t')) \right] \right|_{t'=t_T} \\ + \frac{1}{c_L^2} \left[\oint_{L(t')} \frac{R_i R_j}{R^2} \frac{V_k}{R - \mathbf{R} \cdot \mathbf{V}/c_L} dL_n(\mathbf{s}(t')) \right] \right|_{t'=t_L} \\ + \int_{1/c_L}^{1/c_T} \left[\oint_{L(t')} \left(\frac{3R_i R_j}{R^2} - \delta_{ij} \right) \frac{\kappa V_k}{R - \kappa \mathbf{R} \cdot \mathbf{V}} dL_n(\mathbf{s}(t')) \right] \right|_{t'=t_\kappa} d\kappa \Big\}, \end{aligned} \quad (108)$$

where $\mathbf{V} = \mathbf{V}(t')$, $\mathbf{R} = \mathbf{R}(t')$, $\mathbf{s}(t')$ and $L(t')$ have to be computed at the corresponding retarded times. Thus, the Liénard-Wiechert tensor potentials of a dislocation loop (107) and (108) are line integrals of the Liénard-Wiechert tensor potentials of a dislocation point source (49) and (50) integrated over three loop curves $L(t_T)$, $L(t_L)$, and $L(t_\kappa)$ at the corresponding retarded times and with the integrands $dL_n(\mathbf{s}(t_T))$, $dL_n(\mathbf{s}(t_L))$, and $dL_n(\mathbf{s}(t_\kappa))$. Eqs. (107) and (108) are retarded line integrals. Thus, the Liénard-Wiechert tensor potentials of a dislocation loop are the line integrals of a ‘dislocation point source’ acting on $\mathbf{s}(t')$ integrated over the dislocation loop $L(t')$ and evaluated at the corresponding retarded times. The elastic fields of a non-uniformly moving dislocation loop are obtained by substituting Eqs. (107) and (108) into (94) and (95) reproducing the formulae (52) and (53). The fields produced by a dislocation loop can therefore be computed by first determining the Liénard-Wiechert tensor potentials (107) and (108) and then obtaining the elastic fields by means of the relations (94) and (95). More directly, the Liénard-Wiechert tensor potentials of a dislocation loop (107) and (108) can be obtained by substituting the dislocation density tensor and Mura’s dislocation velocity tensor of a dislocation loop

$$\alpha_{ij}(\mathbf{r}, t) = b_i \oint_{L(t)} \delta(\mathbf{r} - \mathbf{s}(t)) dL_j(\mathbf{s}(t)), \quad (109)$$

$$V_{kij}(\mathbf{r}, t) = b_i \oint_{L(t)} V_k(t) \delta(\mathbf{r} - \mathbf{s}(t)) dL_j(\mathbf{s}(t)), \quad (110)$$

into Eqs. (96) and (97) or in Eqs. (103) and (104). Eqs. (109) and (110) can be read-off from Eqs. (33) and (34), using the relations (11) and (100).

In the static case, the Liénard-Wiechert tensor potentials of a dislocation loop (107) and (108) reduce to

$$\phi_{ijmn}(\mathbf{r}) = \frac{b_m}{16\pi\mu(1-\nu)} \oint_L \frac{1}{R} \left[(3-4\nu)\delta_{ij} + \frac{R_i R_j}{R^2} \right] dL'_n, \quad (111)$$

and $A_{ijkmn}(\mathbf{r}) = 0$, where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

Comparing Eqs. (94) and (95) with (65) and (66), we obtain for the two-dimensional Liénard-Wiechert tensor potentials of plane strain

$$\phi_{ijmz} = \phi_{ij} b_m \ell_z, \quad A_{ijkmz} = A_{ijk} b_m \ell_z, \quad (112)$$

and with (70) and (71), we obtain for the two-dimensional Liénard-Wiechert tensor potentials of anti-plane strain

$$\phi_{zzzz} = \phi_{zz} b_z \ell_z, \quad A_{zzkzz} = A_{zzk} b_z \ell_z. \quad (113)$$

From the condition (102), we get

$$\dot{\phi}_{ij} + A_{ijk,k} = 0 \quad (114)$$

and

$$\dot{\phi}_{zz} + A_{zzk,k} = 0. \quad (115)$$

A ‘gauge’ condition like (115) was already mentioned by Lardner [1].

Therefore, it is obvious that the elastodynamical Liénard-Wiechert tensor potentials can be obtained from Mura’s dislocation tensor potentials. Especially, for a dislocation loop the Mura dislocation tensor potentials give directly the Liénard-Wiechert tensor potentials as retarded line integrals along loop curves computed at the corresponding retarded times.

9 Conclusions

In this work, we have investigated the fundamentals of the non-uniform motion of dislocations based on electromagnetic analogies. We have examined and solved to following items:

- retarded elastic fields/Jefimenko type equations for dislocation fields
- retarded dislocation tensor potentials
- 3-D Liénard-Wiechert tensor potentials of point dislocation sources
- 3-D Liénard-Wiechert tensor potentials of a dislocation loop
- Heaviside-Feynman type equations for a dislocation loop

- 2-D Liénard-Wiechert tensor potentials of straight dislocations
- singularities of the near-fields of straight dislocations.

This analysis provides the treatment of general dislocation motion (with inertia effects) in terms of retarded fields. We have introduced the elastodynamic Liénard-Wiechert tensor potentials as fundamental quantities for the non-uniform motion of dislocations. For the general motion of a dislocation loop, the solution is obtained in terms of Liénard-Wiechert tensor potentials depending on the retarded times and in a more closed form as line integrals (Heaviside-Feynman type formulae). The Jefimenko type and Heaviside-Feynman type equations contain only time derivatives and no spatial derivatives. Thus, for the general motion of a loop the solution is evaluated in a closed form. It is concluded that causal dependencies of the elastic fields and Liénard-Wiechert tensor potentials of a non-uniformly moving dislocation loop are described by retarded line integrals along the dislocation loop. The non-uniform motion of dislocations leads to retardation effects which are important in the interaction between dislocations. Thus, the interaction between dislocations during the dislocation motion is not instantaneous, without being limited by the speed of the sound waves. We think that these results will open a new window for a better understanding of the non-uniform motion of dislocation fields. Especially, they can be used in simulations of discrete dislocation dynamics. Retardation effects should be considered in dislocation dynamics simulations at high strain rates and they become important at extremely high frequencies.

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